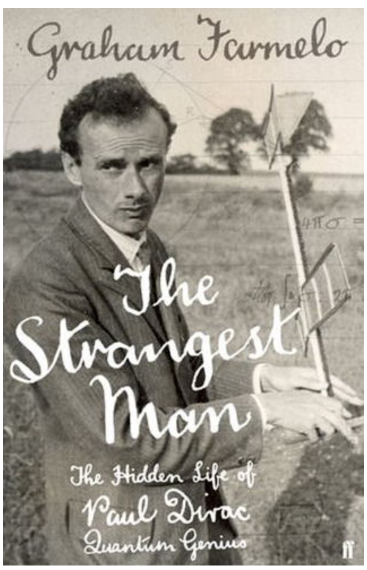
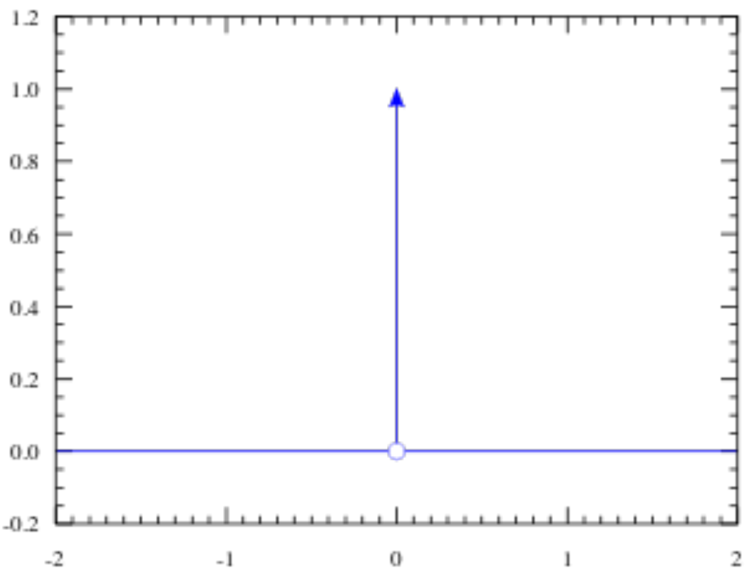
Mathematical preliminaries

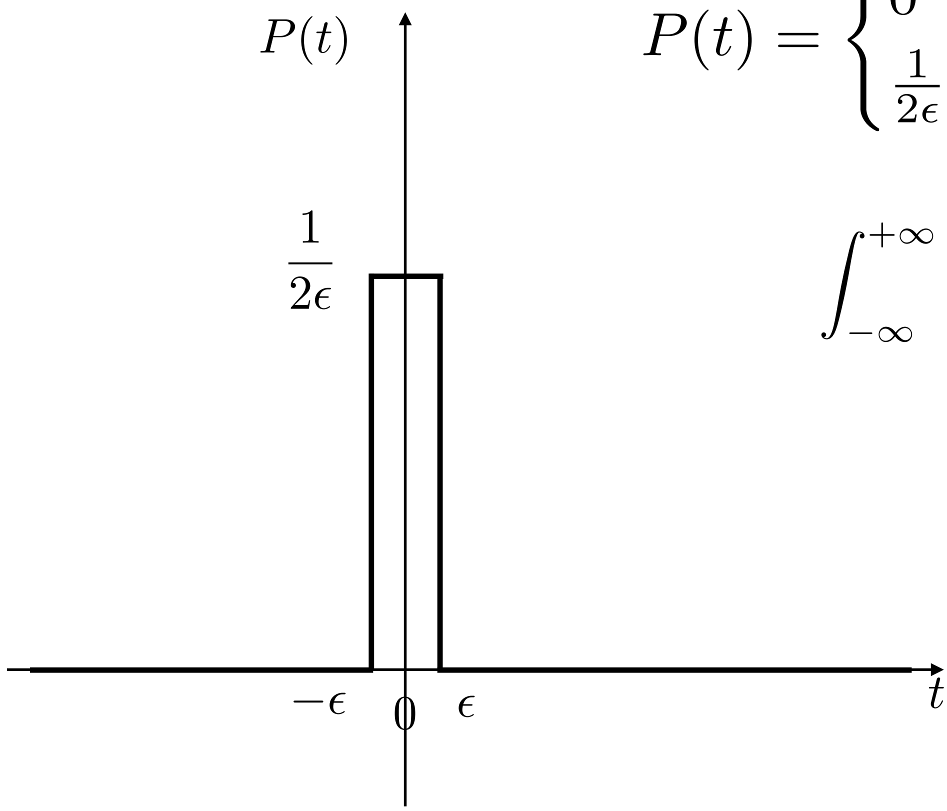
- The Dirac's Delta-function
- The Convolution Integral
- Sampling of a function
- Linear, first-order, differential equations
- Numerical solutions

Dirac's Delta Function

$$\delta(x)$$

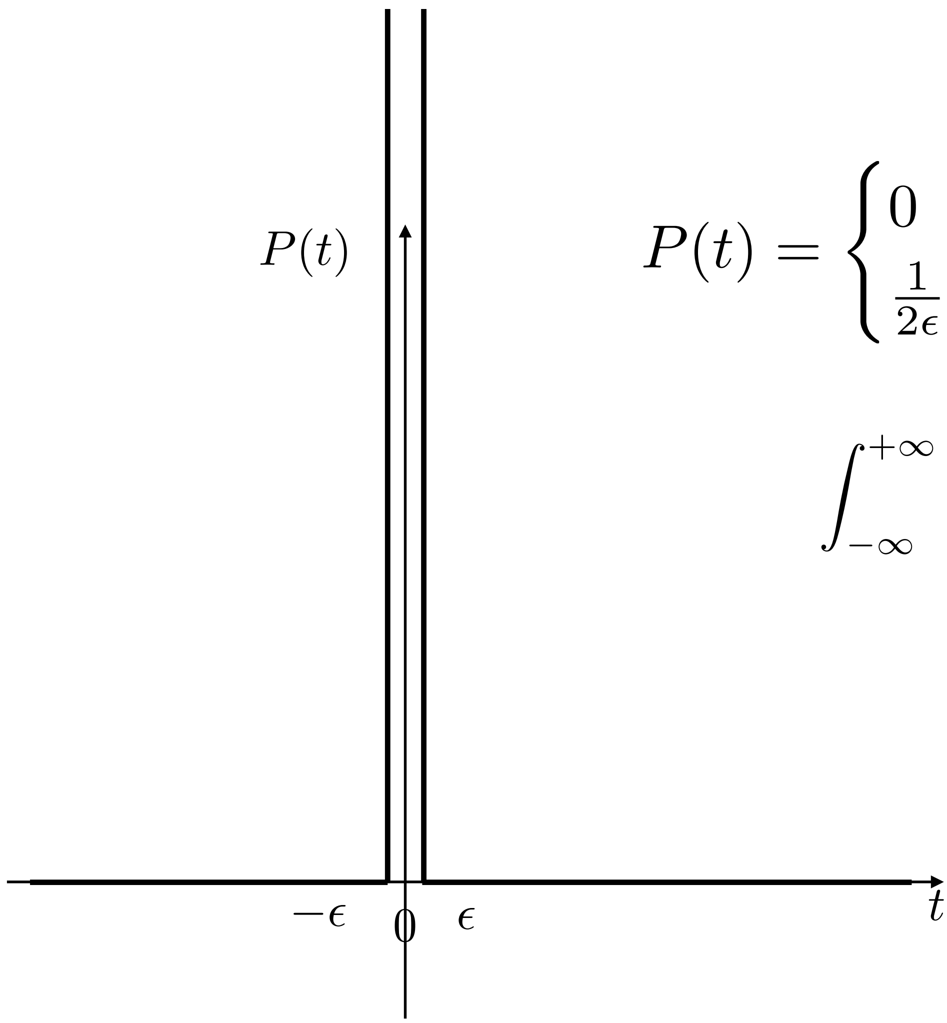






$$P(t) = \begin{cases} 0 & \text{if } |t| > \epsilon \\ \frac{1}{2\epsilon} & \text{if } |t| < \epsilon \end{cases}$$

$$\int_{-\infty}^{+\infty} P(t)dt = 1$$



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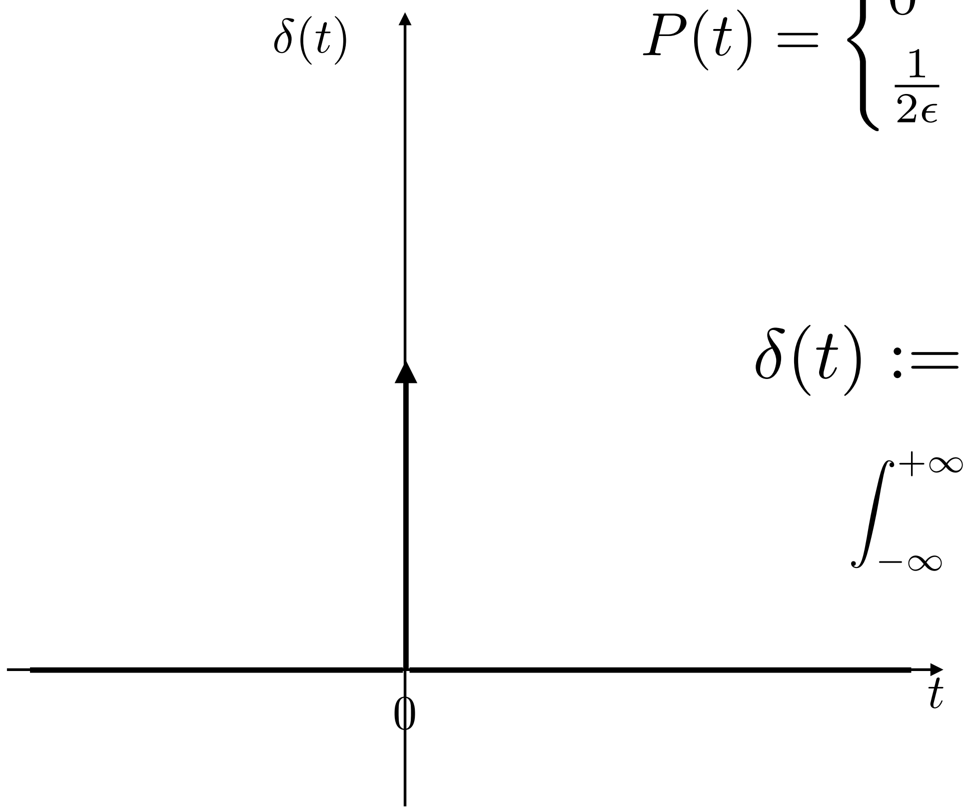
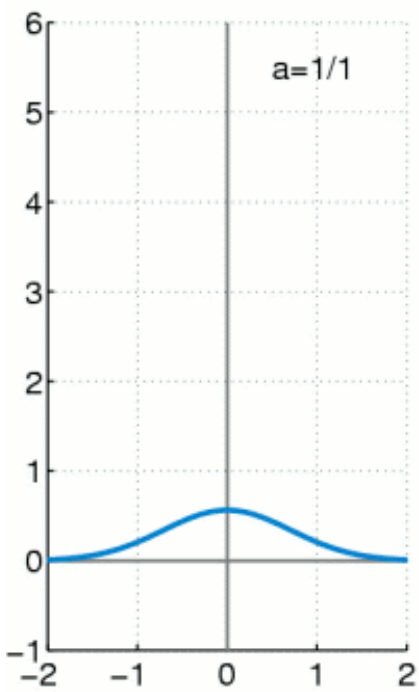


$$P(t) = \begin{cases} 0 & \text{if } |t| > \epsilon \\ \frac{1}{2\epsilon} & \text{if } |t| < \epsilon \end{cases}$$

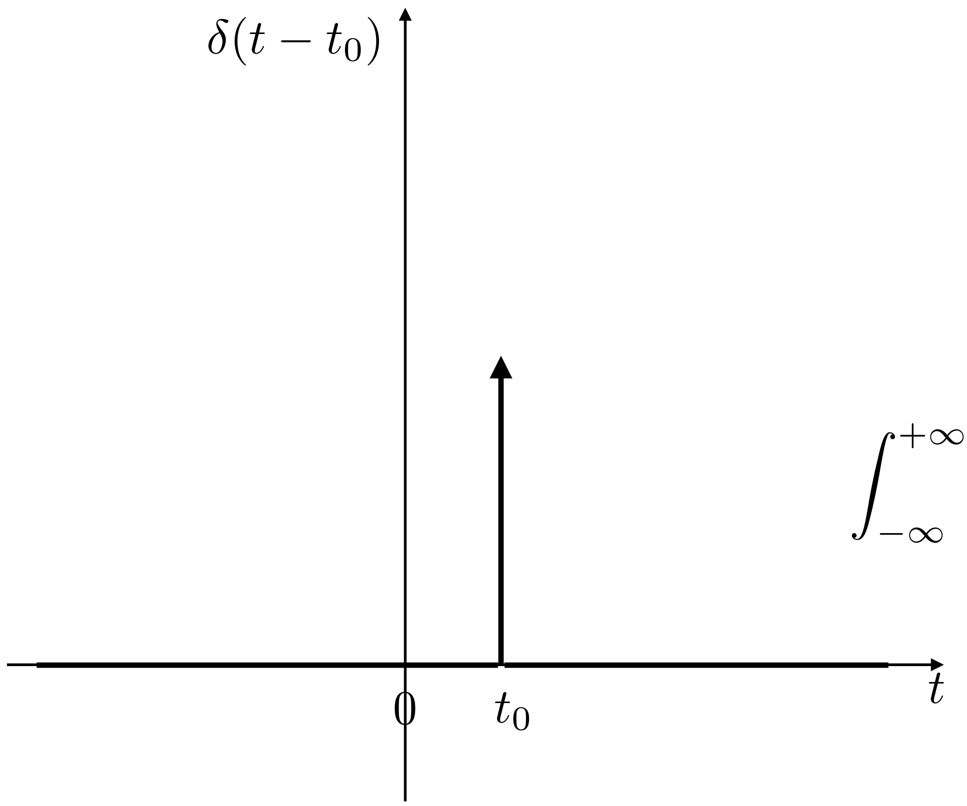
$$\delta(t) := \lim_{\epsilon \to 0} P(t)$$

$$\int_{-\infty}^{+\infty} \delta(t) dt = 1$$

$$\delta(x) := \lim_{\varepsilon \to 0} \frac{1}{\sqrt{2\pi\epsilon^2}} e^{-\frac{x^2}{2\epsilon^2}}$$



$$\int_{-\infty}^{+\infty} \delta(t) dt = 1$$



$$\int_{-\infty}^{+\infty} \delta(t-t_0)dt = 1$$

Dirac's Delta Function

$$\int\limits_{-\infty}^{+\infty} \delta(x)dx = 1 \qquad \int\limits_{-\infty}^{+\infty} \delta(t)dt = 1 \qquad \int\limits_{-\infty}^{+\infty} \delta(t \pm t_0)dt = 1$$

$$\delta(-x) = \delta(x)$$

$$\delta(t-t_0) = \delta(t_0-t)$$

$$\int\limits_{0}^{100} \delta(t+32)dt =$$

$$\int\limits_{-\infty}^{+\infty} g(x)\delta(x)dx = g(0)$$

$$\int\limits_{-\infty}^{+\infty} g(x)\delta(x-x_0)dx = g(x_0)$$

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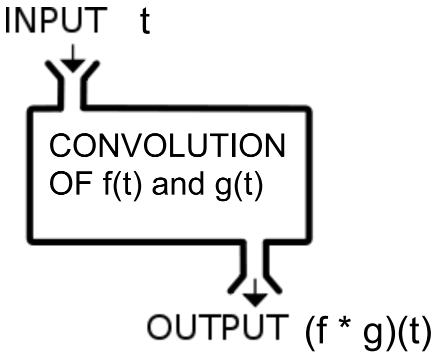
The "*convolution* (integral)" of two functions is… a function

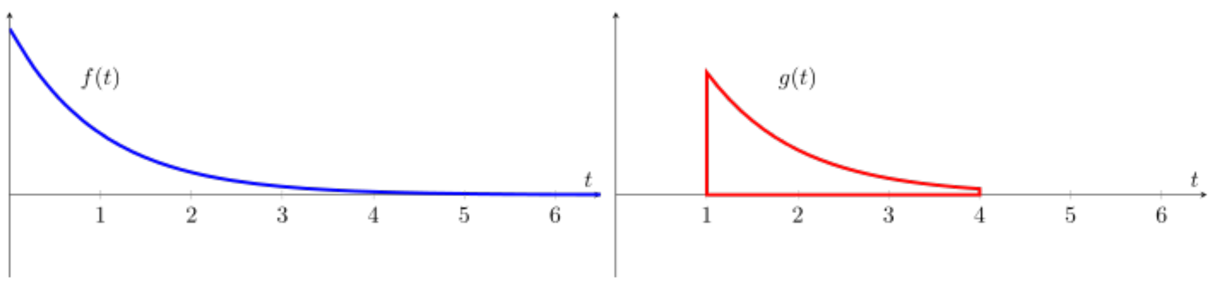


$$f(t) \quad g(t)$$

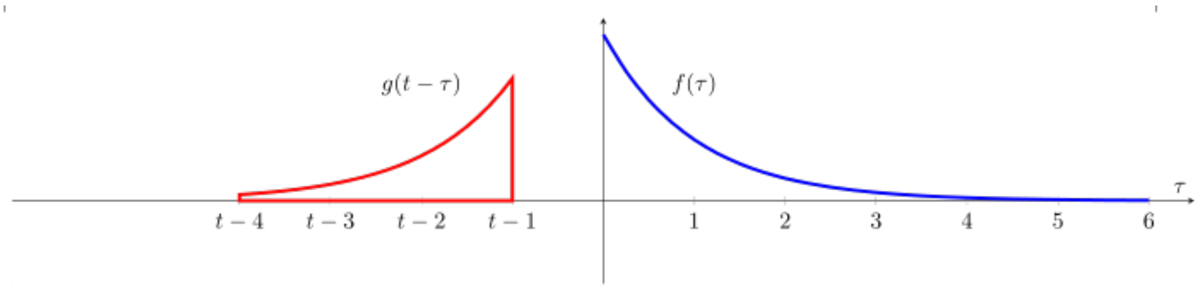
$$(f * g)(t) := \int_{-\infty}^{+\infty} f(\tau) g(t - \tau)\, d\tau$$

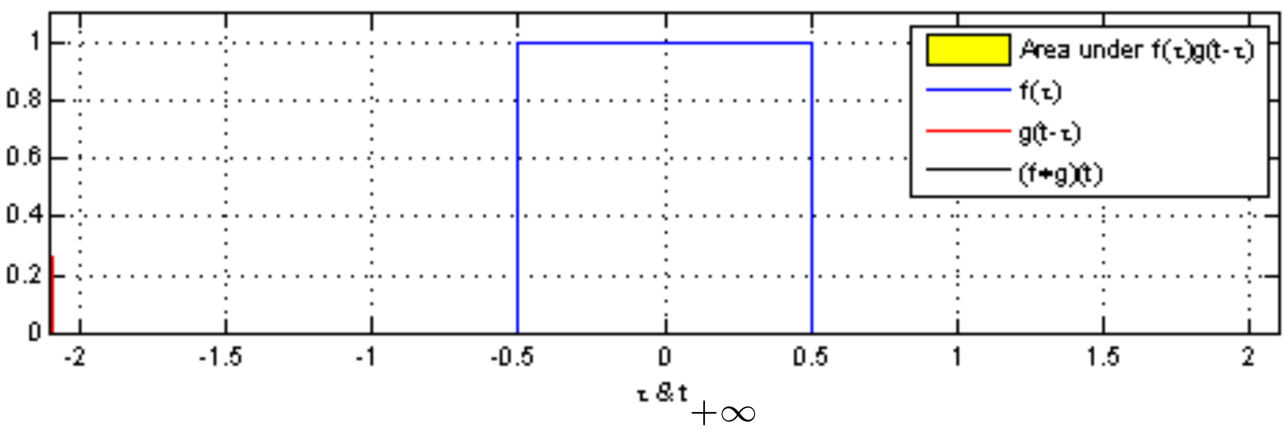
$$f * g = g * f$$

$$(f * g)(t) := \int_{-\infty}^{+\infty} f(\tau) g(t - \tau)\, d\tau$$



$$(f * g)(t) := \int_{-\infty}^{+\infty} f(\tau)g(t-\tau)\,d\tau$$

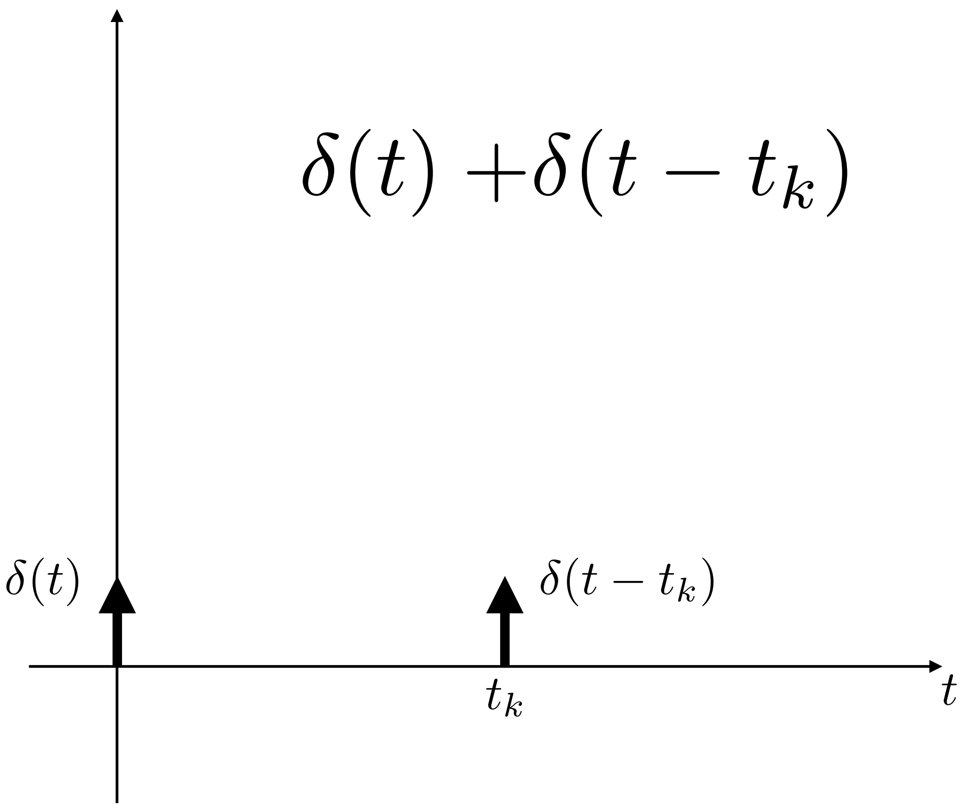




$$(f * g)(t) := \int_{-\infty}^{+\infty} f(\tau)g(t-\tau)\,d\tau$$

Sum of two Dirac's Delta Functions

$$\delta(t) + \delta(t - t_k)$$



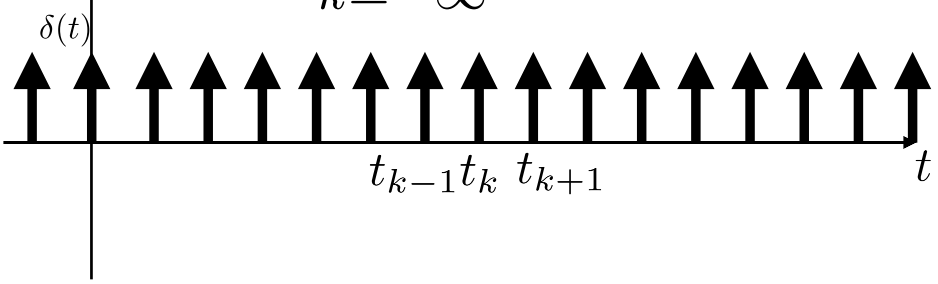
A *train* of Dirac's Delta Functions

$$\ldots, 0, t_1, t_2, t_3, \ldots, t_k, t_{k+1}, \ldots$$

$$t_{k+1} - t_k = \Delta t$$

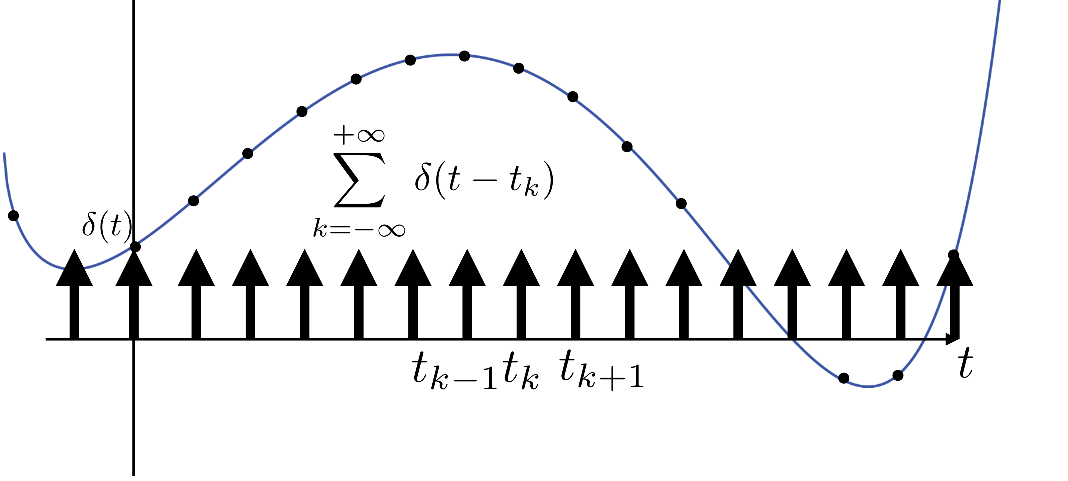
$$\ldots + \delta(t) + \delta(t-t_1) + \delta(t-t_2) + \delta(t-t_3) + \ldots + \delta(t-t_k) + \delta(t-t_{k+1}) + \ldots$$

$$\sum_{k=-\infty}^{+\infty} \delta(t-t_k) = \sum_{k=-\infty}^{+\infty} \delta(t-k\Delta t)$$



Sampling of a Function: <u>discrete</u> case

$$f(t) \sum_{k=-\infty}^{+\infty} \delta(t-t_k) = \sum_{k=-\infty}^{+\infty} f(t_k)\delta(t-t_k)$$



Sampling of a Function: continuous case

$$f(t)\sum_{k=-\infty}^{+\infty}\delta(t-t_k) = \sum_{k=-\infty}^{+\infty} f(t_k)\delta(t-t_k)$$

$$f(t)*\delta(t) = \int_{-\infty}^{+\infty} f(\tau)\delta(t-\tau)d\tau = f(t)$$

Mathematical preliminaries

- The Dirac's Delta-function
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- Sampling of a function
- **Linear**, **first-order**, *differential* equations
- Numerical solutions

(Algebraic) equations

$$? \; x \; : \; f(x) = 0$$

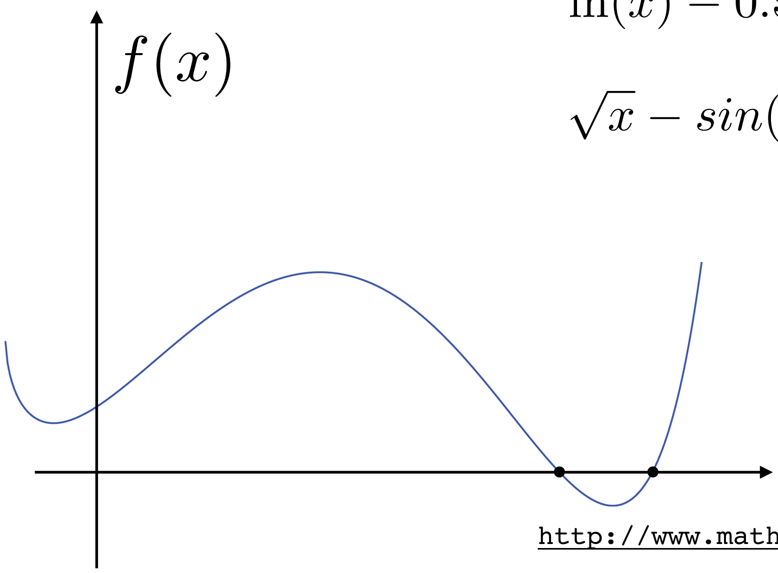
$$x + 5 = 0$$

$$ax^2 + bx + c = 0$$

Robert Recorde
(1551-1558)

$$\ln(x) - 0.5x = -0.5$$

$$\sqrt{x} - sin(\frac{1}{x}) = 0$$



$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

http://www.mathsisfun.com/data/function-grapher.php

Ordinary *differential* equations (o.d.e)

$$? \; f(x) \; :$$

$$\frac{d}{dx}f(x) = -f(x) \qquad \frac{d}{dx}f(x) = 2f(x) \qquad \frac{d}{dx}f(x) = \frac{1}{x}$$

$$\frac{d}{dx}f(x) = -0.1f(x) + sin(x) \qquad \frac{d}{dx}f(x) = -0.1f(x) + 42$$

$$f(x) = -41??$$

$$f(x) = sin(x)?? \qquad f(x) = ln(x)?? \qquad f(x) = sin^2(x)??$$

$$f(x) = e^{-x}?? \qquad f(x) = e^{2x}?? \qquad f(x) = e^{-x} + 42 \; ??$$

Ordinary *differential* equations (o.d.e)

$$\frac{d}{dx} f(x) = f(x)^2 + \frac{\sqrt{sin(f(x))}}{x} - 2x^8$$

$$\frac{d}{dx} f(x) = G(f(x))$$

$$\frac{d}{dx} f(x) = G(f(x), x)$$

The only o.d.e. you <u>must</u> study

$$\frac{d}{dx} f(x) \ = \ a \ f(x)$$ with a constant

homogeneous, first order, with constant coefficient(s)

$$f(x) = K \ e^{ax} \qquad \frac{d}{dx} \left(K \ e^{ax}\right) = K \ a \ e^{ax}$$

The only o.d.e. you <u>must</u> study

$$\frac{d}{dx} f(x) \;=\; a\; f(x)$$ with a constant

$$f(x_0) = f_0$$ with an *initial* condition

homogeneous, first order, with constant coefficient(s)

$f(x) = f_0 \; e^{-ax0} \; e^{ax}$ $f(x) = f_0 \; e^{a(x-x_0)}$

The only o.d.e. you <u>must</u> study

$$\frac{d}{dx} f(x) \;=\; a\; f(x)$$

$$f(x_0) = f_0$$

So.. what is the solution? Does the solution exist? Is it single?

This is known as *Cauchy's problem* and a math theorem insures that

the solution exists and it is unique

Why bothering with o.d.e

In (bio)physics, that o.d.e. explain or approximate many phenomena, generally occurring (spontaneously) as a function of time, without any external driving influence..

If you solve that o.d.e., you can make "predictions" on the future (i.e. evaluate *f(t)* for any *t*, also for *t* = …1st Jan 2239)

the voltage across a capacitor during spontaneous discharging;

the temperature of an object, exchanging heat by thermal conduction;

the level of water in a leaky pool (e.g. with a hole on its bottom);

the concentration of a substance in a given point (e.g. ink in water);

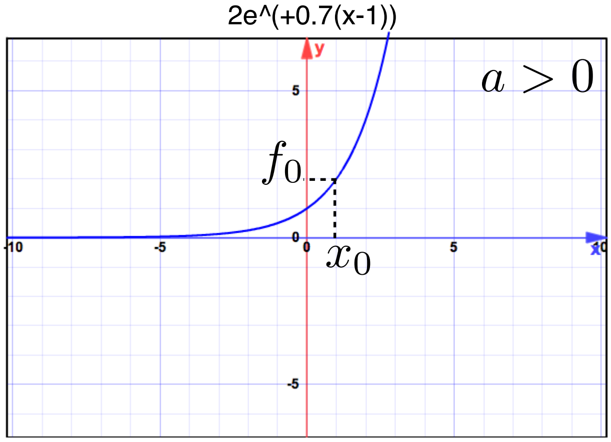
the sodium ions, exchanged across the membrane of a neuron;

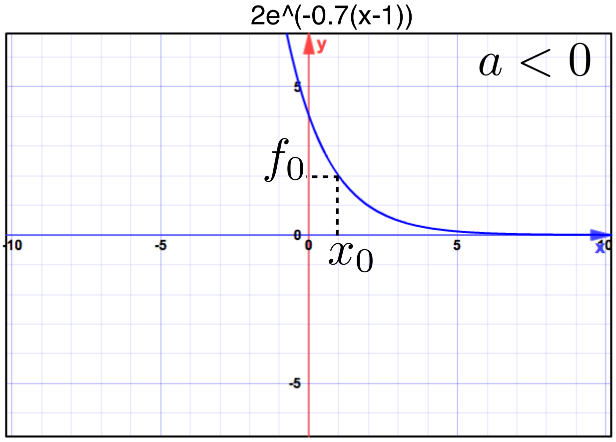
the electrical potential across cell membranes;

The only o.d.e. you <u>must</u> study

$$\frac{df(x)}{dx} = af(x) \qquad f(x_0) = f_0$$

$$\boxed{f(x) = f_0 e^{a(x-x_0)}}$$





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Separation of variables

$$\frac{df(x)}{dx} = af(x) \qquad f(x_0) = f_0$$

$$\boxed{f(x) = f_0 e^{a(x-x_0)}}$$

$$\frac{df(x)}{f(x)} = a\ dx \qquad \int \frac{df(x)}{f(x)} = \int a\ dx \qquad \int \frac{df}{f} = \int a\ dx$$

$$ln(f) + constant = ax + constant$$

$$ln(f) = ax + H$$

$$e^{ln(f)} = e^{ax+H}$$

$$f(x) = e^{ax}e^{H}$$

$$f(x) = Ke^{ax}$$

Separation of variables

$$\frac{df(x)}{dx} = af(x) \qquad f(x_0) = f_0$$

$$\boxed{f(x) = f_0 e^{a(x-x_0)}}$$

$$\frac{df(x)}{f(x)} = a\ dx \qquad \cancel{\int \frac{df(x)}{f(x)} = \int a\ dx} \qquad \int_{f_0}^{f} \frac{df}{f} = \int_{x_0}^{x} a\ dx$$

$$ln(f) - ln(f_0) = a(x - x_0)$$

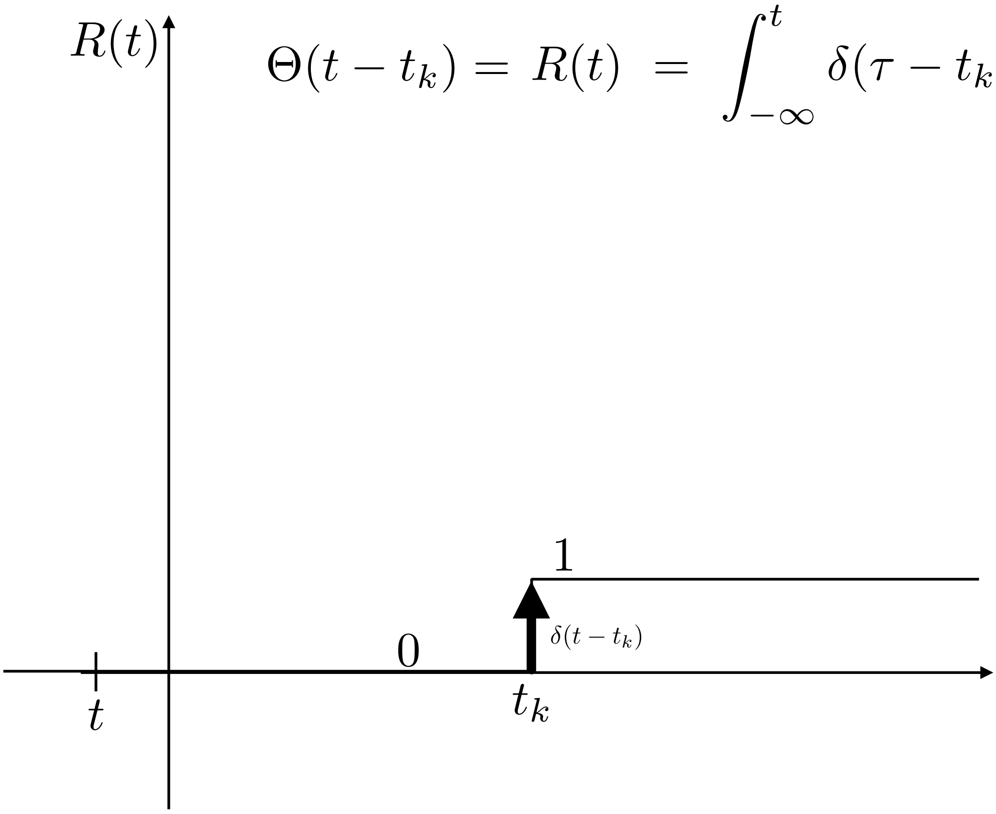
$$ln(f/f_0) = a(x - x_0)$$

$$e^{ln(f/f_0)} = e^{a(x-x_0)}$$

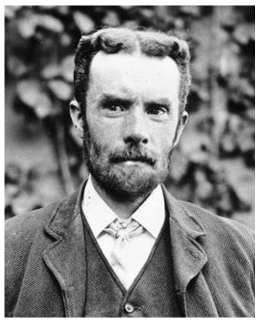
$$f(x)/f_0 = e^{a(x-x_0)}$$

Integrating a Dirac's Delta

$$\Theta(t - t_k) = R(t) \; = \int_{-\infty}^{t} \delta(\tau - t_k)d\tau$$



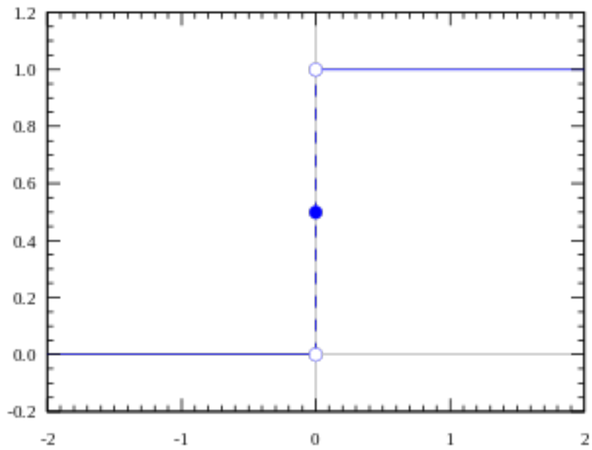
Heaviside's "step" function


Oliver Heaviside
(1850 - 1925)

$$\Theta(x) := \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \end{cases}$$

$$\Theta(x) = \int_{-\infty}^{x} \delta(\tau)d\tau$$

$$\frac{d}{dx}\Theta(x) = \delta(x)$$



$$\Theta(x - x_0) := \begin{cases} 0 & \text{for } x < x_0 \\ 1 & \text{for } x > x_0 \end{cases}$$

The inhomogeneous case:

the principle of *superposition of effects*

$$\frac{df(t)}{dt} = af(t) + g_1(t)$$

$$f_1(t)$$

$$\frac{df(t)}{dt} = af(t) + g_2(t)$$

$$f_2(t)$$

$$\frac{df(t)}{dt} = af(t) + (\alpha g_1(t) + \beta g_2(t))$$

$$\alpha f_1(t) + \beta f_2(t)$$

The inhomogeneous case:

solution of the homogenous o.d.e.
+ "particular" solution

$$\frac{df(t)}{dt} = af(t)$$

$$f(t) = Ke^{at}$$

$$\frac{df(t)}{dt} = af(t) + u$$

$$f_p(t)$$

$$\frac{df(t)}{dt} = af(t) + (0 + u) \qquad f(t_0) = f_0$$

$$f(t) = Ke^{at} + f_p(t)$$

The inhomogeneous case:

constant "external" input *u*

$$\frac{df(t)}{dt} = af(t) + u \qquad\qquad f(t_0) = f_0$$

$$\boxed{f_p(t)}$$

<u>Heuristics</u>: since 'u' is constant, maybe… $f_p(t)$ is also constant.

$$f_p(t) = c \qquad \frac{df_p(t)}{dt} = 0 = a\ c\ +\ u \qquad c = -\frac{u}{a}$$

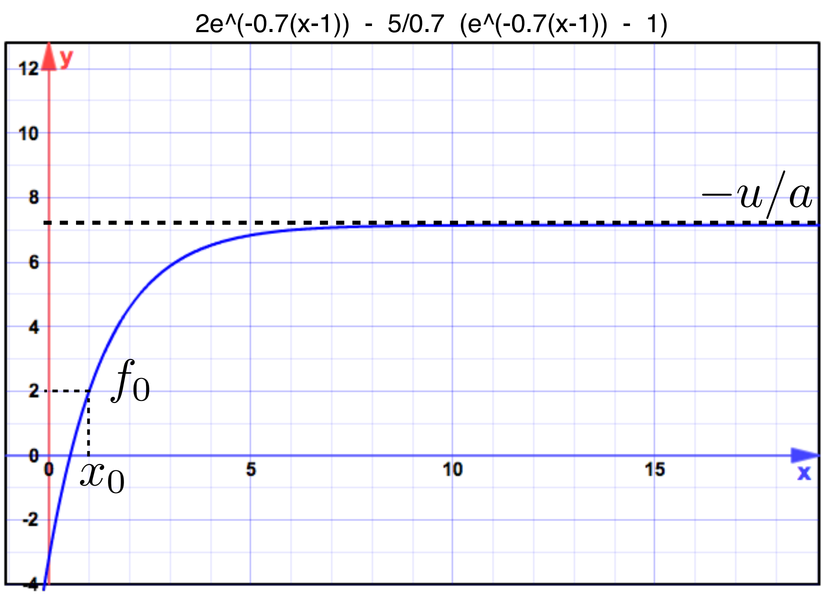
$$f(t) = Ke^{at} - \frac{u}{a} \qquad f_0 = Ke^{at_0} - \frac{u}{a} \qquad K = f_0 e^{-at_0} + \frac{u}{a}e^{-at_0}$$

$$\boxed{f(t) = f_0 e^{a(t-t_0)} + \frac{u}{a}\left(e^{a(t-t_0)} - 1\right)}$$

The inhomogeneous case:

constant "external" input *u*

$$\boxed{f(t) = f_0 e^{a(t-t_0)} + \frac{u}{a}\left(e^{a(t-t_0)} - 1\right)}$$



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Digression

steady-state (equilibrium) of the solution

$$\frac{df(t)}{dt} = af(t) + u$$

$$\lim_{t \to +\infty} f(t) = f_\infty$$

$$\frac{d}{dt} f_\infty \;=\; 0$$

$$0 = af_\infty \; + u$$

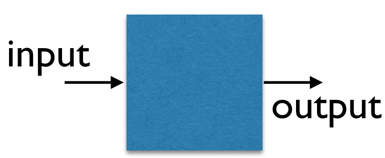
$$f_\infty \;= -u/a$$

The inhomogeneous case:

no “external” input *u*

$$\frac{df(t)}{dt} \;= -f(t)$$

$$f(0) = 1$$

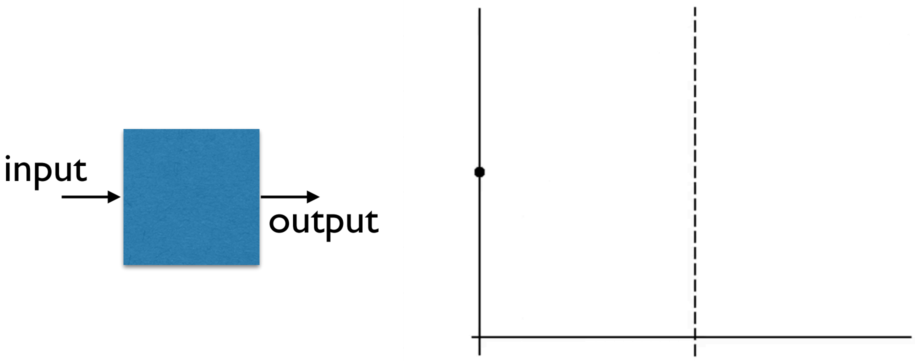


$$f(t) = e^{-t}$$

The inhomogeneous case:

no "external" input until a certain time

$$\frac{df(t)}{dt} = -f(t) \qquad f(0) = 1$$

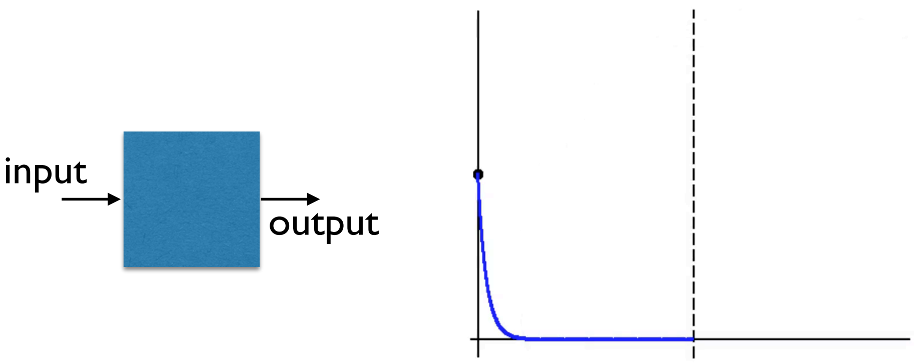


$$f(t) = e^{-t}$$

The inhomogeneous case:

(piece-wise) constant "external" input

$$\frac{df(t)}{dt} = -f(t) + 1.5\Theta(t - t_{on}) \qquad f(0) = 1$$

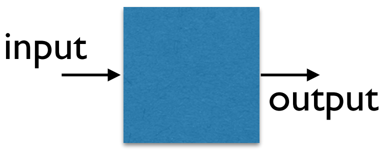


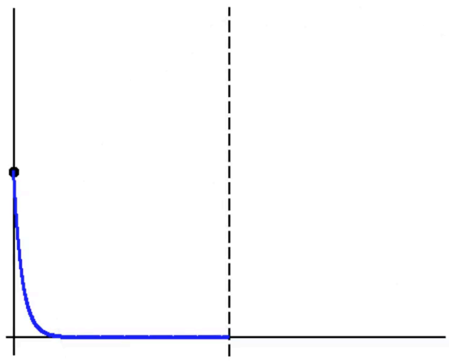
$$f(t) = e^{-t} - 1.5\left(e^{-(t-t_{on})} - 1\right)\Theta(t - t_{on})$$

The inhomogeneous case:

impulse as "external" input

$$\frac{df(t)}{dt} = -f(t) + 1.5\delta(t - t_{on}) \qquad f(0) = 1$$





$$f(t) = e^{-t} + 1.5e^{-(t-t_{on})}\Theta(t - t_{on})$$

The inhomogeneous case:

solution of the homogenous o.d.e.
+ "particular" solution in the zero-state

$$\frac{df(x)}{dx} = af(x) + \delta(x - x_{on}) \qquad f(x_0) = 0$$

$$\boxed{f_p(x) = h(x)}$$

$$h(x) = e^{a(x-x_{on})}\Theta(x - x_{on})$$

Response to the impulse

$$\begin{aligned}\frac{dh(x)}{dx} &= ae^{a(x-x_{on})}\Theta(x - x_{on}) + e^{a(x-x_{on})}\delta(x - x_{on}) \\ &= ae^{a(x-x_{on})}\Theta(x - x_{on}) + \delta(x - x_{on})\end{aligned}$$

The inhomogeneous case:

the “particular” solution, in the most general case

$$\frac{df(t)}{dx} = af(t) + u(t)\Theta(t - t_{on}) \qquad f(t_0) = f_0$$

$$\boxed{f(t) = f_0 e^{a(t-t_0)} + f_p(t)}$$

$$u(t)\Theta(t - t_{on}) = \int_{-\infty}^{+\infty} u(\tau)\Theta(\tau - t_{on})\delta(t-\tau)d\tau$$

$$f_p(t) = \int_{-\infty}^{+\infty} u(\tau)\Theta(\tau - t_{on})h(t-\tau)d\tau$$

$$\boxed{f_p(t) = h(t) * u(t)\Theta(t - t_{on})}$$

First-order, linear, o.d.e. with initial condition

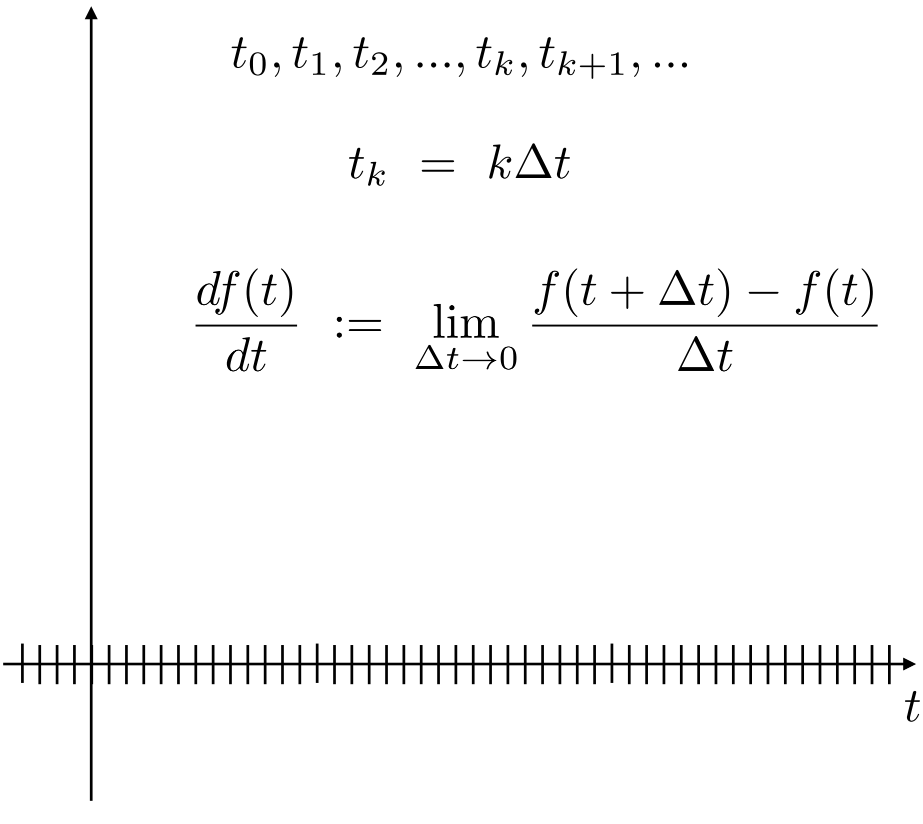
- The homogeneous case
  - The separation of variables method of solution
- The inhomogeneous case
  - The principle of superposition of effects (linearity)
  - Heaviside’s *step* function
  - solution for constant inhomogeneous terms (external inputs)
  - equilibrium (or steady-state) and convergence to it
  - o.d.e. as input-output, filters: step and impulse responses
  - solution for arbitrary inhomogeneous terms (external inputs)

Numerical solutions of an o.d.e.

$$t_0, t_1, t_2, ..., t_k, t_{k+1}, ...$$

$$t_k \;=\; k\Delta t$$

$$\frac{df(t)}{dt} \;:=\; \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$



Numerical solutions of an o.d.e.

explicit Euler's method

$$\frac{df(t)}{dt} = af(t) + g_1(t) \qquad \frac{df(t)}{dt} \approx \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

$$f(0) = f_0$$

$$\frac{f(t+\Delta t) - f(t)}{\Delta t} \approx af(t) + g_1(t)$$

$$f(t+\Delta t) \approx (1 + a\Delta t)\, f(t) + \Delta t\ g_1(t)$$

$$\boxed{\tilde{f}_{k+1} = \tilde{a} f_k + \tilde{g}_k} \qquad \tilde{f}_k \approx f(k\Delta t)$$

$$\tilde{f}_0 = f_0$$

$$\tilde{a} = (1 + a\Delta t)$$

$$g_k = \Delta t g_1(k\Delta t)$$

Numerical solutions of an o.d.e.

explicit Euler's method

$$\frac{df(t)}{dt} = f(t)^2 + lnf(t) \qquad \frac{df(t)}{dt} \approx \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

$$f(0) = f_0$$

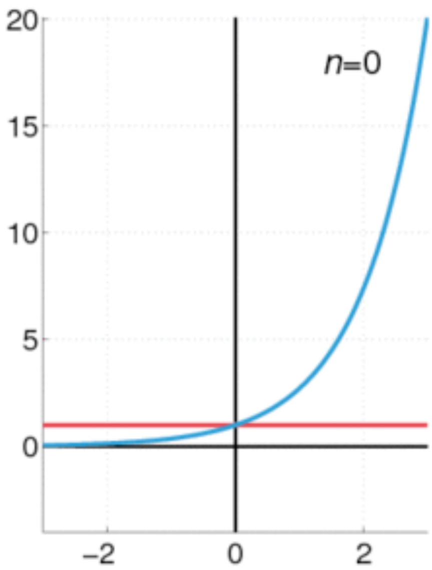
$$\boxed{\tilde{f}_{k+1} = \tilde{f}_k + \Delta t \left(\tilde{f}_k^2 + \ln \tilde{f}_k\right)} \qquad \tilde{f}_k \approx f(k\Delta t)$$

$$\frac{df(t)}{dt} = f(t)^2 + lnf(t) + \frac{a}{t} - f(t)^t$$

$$\boxed{\tilde{f}_{k+1} = \tilde{f}_k + \Delta t \left(\tilde{f}_k^2 + \ln \tilde{f}_k + \frac{a}{k\Delta t} - \tilde{f}_k^{k\Delta t}\right)}$$

$$\tilde{f}_0 = f_0$$

Taylor's polynomial-series expansion



$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots.$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

Analytical solution: iterative expressions

$$\frac{df(t)}{dt} = af(t) + u$$

$$f(t) = f_0 e^{a(t-t_0)} + \frac{u}{a}\left(e^{a(t-t_0)} - 1\right)$$

$$f(t+\Delta t) = f_0 e^{a\Delta t} e^{a(t-t_0)} + \frac{u}{a}\left(e^{a\Delta t} e^{a(t-t_0)} - 1\right)$$

$$f(t+\Delta t) = e^{a\Delta t} f(t) - \frac{u}{a}\left(1 - e^{a\Delta t}\right)$$

$$f_{k+1} = e^{a\Delta t} f_k - \frac{u}{a}\left(1 - e^{a\Delta t}\right)$$

$$e^{a\Delta t} \approx 1 + a\Delta t$$

$$\frac{u}{a}(1 - e^{a\Delta t}) \approx -u\Delta t$$

$$\tilde{f}_{k+1} = \tilde{a} f_k + u\Delta t$$

$$\tilde{a} = (1 + a\Delta t)$$